

# Restricted Approximation by Strongly Sign-Regular Kernels: The Finite Bang-Bang Principle

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## 1. INTRODUCTION AND DESCRIPTION OF THE RESULT

Let  $K(s, t)$  be a real valued function defined on  $S \times T$  where  $S = (a, b)$  and  $T = (c, d)$  are bounded intervals of the real line. More exact requirements on  $K(s, t)$  will be specified later. In this paper we consider the problem of best uniform approximation of a continuous function  $z(s)$  on  $S$  by functions of the form

$$y(s) = \int_T K(s, t) u(t) dt, \quad (1.1)$$

where  $u(t)$  is subject to the inequality constraint

$$|u(t)| \leq 1 \quad \text{for all } t \text{ in } T. \quad (1.2)$$

Thus we wish to minimize

$$\max_{s \in S} \left| z(s) - \int_T K(s, t) u(t) dt \right| \quad (1.3)$$

under the restriction (1.2).

Problems of this type arise in control theory where (1.1) defines a control operator with  $u(t)$  as control function and  $z(s)$  as the "desired state" (cf. [2]).

Our main result is the following. We show that under suitable assumptions on the kernel  $K(s, t)$  there is a unique solution  $\bar{u}$  of the restricted approximation problem such that  $\bar{u}$  is *finitely bang-bang*:

$$\bar{u}(t) = \epsilon(-1)^r \quad \text{for } t \in [t_{r-1}, t_r); \quad r = 1, \dots, n+1, \quad (1.4)$$

for some  $\epsilon = +1$  or  $-1$ , some  $n \geq 1$  and some partition

$$c = t_0 < t_1 < \cdots < t_n < t_{n+1} = d$$

of the interval  $T$ .

In order to obtain such a result we require that  $K(s, t)$  is *strongly sign-regular* (SSR), i.e., that there exists a sequence of numbers  $\epsilon_m$  either  $+1$  or  $-1$  such that

$$\epsilon_m K \begin{pmatrix} s_1, \dots, s_m \\ t_1, \dots, t_m \end{pmatrix} = \epsilon_m \det\{K(s_i, t_k)\} > 0 \quad (1.5)$$

for all  $a < s_1 \cdots s_m < b$ ,  $c < t_1 < \cdots < t_m < d$ . Kernels of this type have well-known *variation diminishing* properties which we use in the proof of our result.

It is possible to apply our theorem to a broad class of control problems, for example to the boundary control of the heat equation. For this class of problems we can derive the finite bang-bang theorem of Karafiat [4] in a much simpler way. A detailed discussion of this result and related applications will be given elsewhere.

## 2. CHARACTERIZATION OF OPTIMAL SOLUTIONS

In order to give a simple presentation of the results we assume that  $K(s, t)$  is continuous on the closure of  $S \times T$ . This ensures that (1.1) defines a continuous linear operator from the space  $L_\infty(T)$  of bounded measurable functions on  $T$  into the space  $C(\bar{S})$  of continuous functions on  $\bar{S} = [a, b]$ . Let us denote this operator by the symbol  $K$ , too:

$$(Ku)(s) = \int_T K(s, t) u(t) dt, \quad s \in S. \quad (2.1)$$

We introduce the following notation:

$$B = \{u \in L_\infty(T) \mid |u(t)| \leq 1 \text{ a.e. on } T\},$$

where we always understand that  $T$  is equipped with ordinary Lebesgue measure.

Now our problem certainly can be formulated as follows:

$$\text{Minimize } \|z - y\|_\infty, \quad y \in K(B), \quad (2.2)$$

where  $z$  is a fixed given element in  $C(\bar{S})$  and  $\|\cdot\|_\infty$  is the uniform norm:

$$\|x\|_\infty = \max_{s \in \bar{S}} |x(s)| \quad \text{for } x \in C(\bar{S}).$$

The following existence result can be proved in a standard way, and we therefore omit the proof (cf. [2]).

LEMMA 1. *There is a  $\bar{y}$  in  $K(B)$  such that*

$$\|z - \bar{y}\|_{\infty} \leq \|z - y\|_{\infty} \quad \text{for all } y \text{ in } K(B).$$

The topological dual of  $C(\bar{S})$  is well known to be isomorphic to the space of measures  $d\alpha$  induced by functions  $\alpha$  on  $\bar{S}$  of bounded variation. We specialize the standard characterization theorem of linear approximation theory (cf. Holmes [3, Sect. 22]) to our case and obtain:

THEOREM 1.  *$\bar{y}$  is solution of (2.2) if and only if there is a measure  $d\alpha$  such that*

$$\int_{\bar{S}} \{z(s) - \bar{y}(s)\} d\alpha(s) = \|z - \bar{y}\|_{\infty}, \quad (2.3)$$

$$\int_{\bar{S}} \bar{y}(s) d\alpha(s) = \max_{y \in K(B)} \int_{\bar{S}} y(s) d\alpha(s). \quad (2.4)$$

Let us draw an important conclusion from this theorem. If  $\bar{y}$  is a solution of our problem, let  $\bar{u} \in B$  be such that

$$\bar{y} = K(\bar{u}).$$

We define

$$\lambda(t) = \int_{\bar{S}} K(s, t) d\alpha(s), \quad t \in T, \quad (2.5)$$

where  $d\alpha$  is the measure appearing in Theorem 1. Using this notation, we write (2.4) in the form

$$\int_T \bar{u}(t) \lambda(t) dt = \max_{u \in B} \int_T u(t) \lambda(t) dt, \quad (2.6)$$

which we call the *maximum principle*. It immediately implies

$$\bar{u}(t) = \text{sgn } \lambda(t) \quad \text{for } \lambda(t) \neq 0. \quad (2.7)$$

Now it is clear that we are going to prove our main result by showing that  $\lambda(t)$  has only a finite number of zeroes on  $T$ . This is the subject of the next section.

## 3. THE FINITE BANG-BANG PRINCIPLE

Let  $d\alpha = d\alpha^+ - d\alpha^-$  be the decomposition of the measure  $d\alpha$  into the difference of two nonnegative measures such that  $\text{supp } d\alpha^+ \cap \text{supp } d\alpha^- = \emptyset$ . Here  $\text{supp}$  denotes the support of a measure.

It is well known (cf. Holmes [3, p. 81, Exercise 33]), that (2.3) implies

$$\begin{aligned} \text{supp } d\alpha^+ \subset P^+ &= \{s \in \bar{S} / z(s) - \bar{y}(s) = \|z - \bar{y}\|_\infty\} \\ \text{supp } d\alpha^- \subset P^- &= \{s \in \bar{S} / z(s) - \bar{y}(s) = -\|z - \bar{y}\|_\infty\}. \end{aligned} \quad (3.1)$$

The following almost trivial lemma gives us the main tool for the proof of our theorem.

LEMMA 2. *Let  $z \neq \bar{y}$ . Then there is a partition*

$$a = s_0 < s_1 < \cdots < s_m < s_{m+1} = b$$

*of  $S$  such that, for each  $S_i = (s_{i-1}, s_i)$ , the intersection of  $S_i$  with one of the sets  $P^+$  or  $P^-$  is empty.*

We do not want to waste space for a proof of this simple lemma which is just a formal statement of the fact that a continuous function on a compact interval cannot oscillate between its absolute extrema infinitely often unless it is constant.

COROLLARY 1. *Let  $\|z - \bar{y}\|_\infty > 0$  where  $\bar{y}$  is a solution of the approximation problem. If  $d\alpha$  is a measure satisfying (2.3), then there exists a partition of the interval  $S$  into finitely many subintervals  $S_1, \dots, S_{m+1}$  such that the sign of the measure induced on the subintervals by  $d\alpha$  alternates; i.e., for any  $y \in C(\bar{S})$*

$$\int_S y(s) d\alpha(s) = \epsilon \sum_{i=1}^{m+1} (-1)^i \int_{S_i} y(s) |d\alpha(s)|,$$

*$\epsilon = +1$  or  $-1$ , where it can be assumed that  $\int_{S_i} d\alpha(s) \neq 0$ ,  $i = 1, \dots, m+1$ .*

The preceding corollary follows immediately from the fact that  $d\alpha \neq 0$  (because of (2.3) and  $\|z - \bar{y}\|_\infty > 0$ ), Eqs. (3.1) and Lemma 1.

If a measure  $d\alpha$  has the alternation property described above, then we write

$$S^-(d\alpha) = m$$

and say that  $d\alpha$  has exactly  $m$  sign changes on  $S$ .

Our aim is now to obtain a bound on the number of zeroes of the function  $\lambda(t)$  defined by (2.5).

DEFINITION 1. (cf. [5, p. 230]). Let  $x = (x_1, \dots, x_n)$  be a vector of real numbers. We denote by  $S^+(x)$  the maximum number of sign changes possible in the vector  $x$  by allowing each zero to be replaced by  $+1$  or  $-1$ . The number  $S^+(f)$  of sign changes of a real valued function  $f(t)$  defined on an ordered subset  $T$  of the real line, is  $\sup S^+(f(t_1), \dots, f(t_n))$ , where the supremum is extended over all sets  $t_1 < t_2 < \dots < t_n$  ( $t_i \in T$ ;  $n$  arbitrary but finite).

It is clear by definition of  $S^+(f)$  that  $S^+(f) < \infty$  implies that  $f$  has only finitely many zeroes.

THEOREM 2. Let  $K(s, t)$  be SSR on  $S \times T$  (cf. (1.5)), and assume that

$$S^-(d\alpha) = m.$$

Then

$$S^+(\lambda) \leq m,$$

where

$$\lambda(t) = \int_S K(s, t) d\alpha(s), \quad t \in T.$$

We do not want to prove this theorem in detail because it seems to belong to the well-known facts on SSR kernels. For sake of completeness we just give the main idea of the proof, modelled after the one given in [5, p. 234], for a related problem.

We define

$$\phi(t, i) = \int_{S_i} K(s, t) |d\alpha(s)| \quad (i = 1, \dots, m+1).$$

Then, for any partition  $t_1 < t_2 < \dots < t_n$ ,  $t_k \in T$ ,

$$\lambda(t_k) = \{\operatorname{sgn} \lambda(S_1)\} \sum_{i=1}^{m+1} (-1)^{i+1} \phi(t_k, i).$$

The composition formula for determinants [5, p. 98] shows that the matrix  $\{\phi(t_k, i)\}$  is SSR. The desired result follows immediately by a theorem on the variation diminishing properties of strongly sign-regular matrices (Gantmacher and Krein [1, p. 285]).

Collecting our results, we come to the final formulation of our theorem announced in the introduction.

THEOREM 3. Let  $\min_{u \in B} \|z - Ku\|_\infty > 0$ . If  $K(s, t)$  is SSR, then any solution  $\bar{u}$  is finitely bang-bang.

COROLLARY 2. There is a unique solution of the restricted approximation problem.

*Proof.* Let  $u_1 \neq u_2$  be solutions. Then

$$\bar{u} = \frac{1}{2}u_1 + \frac{1}{2}u_2$$

is a solution, too, which cannot be bang-bang, in contradiction to Theorem 3 if both  $u_1$  and  $u_2$  have this property.

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